

Problem 2.31

[Difficulty: 4]

2.31 A flow is described by velocity field $\vec{V} = ay^2\hat{i} + b\hat{j}$, where $a = 1 \text{ m}^{-1}\text{s}^{-1}$ and $b = 2 \text{ m/s}$. Coordinates are measured in meters. Obtain the equation for the streamline passing through point (6, 6). At $t = 1 \text{ s}$, what are the coordinates of the particle that passed through point (1, 4) at $t = 0$? At $t = 3 \text{ s}$, what are the coordinates of the particle that passed through point $(-3, 0)$ 2 s earlier? Show that pathlines, streamlines, and streaklines for this flow coincide.

Given: 2D velocity field

Find: Streamlines passing through (6,6); Coordinates of particle starting at (1,4); that pathlines, streamlines and streaklines coincide

Solution:

For streamlines
$$\frac{v}{u} = \frac{dy}{dx} = \frac{b}{a \cdot y^2} \quad \text{or} \quad \int a \cdot y^2 dy = \int b dx$$

Integrating
$$\frac{a \cdot y^3}{3} = b \cdot x + c$$

For the streamline through point (6,6)
$$c = 60 \quad \text{and} \quad y^3 = 6 \cdot x + 180$$

For particle that passed through (1,4) at $t = 0$
$$u = \frac{dx}{dt} = a \cdot y^2 \quad \int 1 dx = x - x_0 = \int a \cdot y^2 dt \quad \text{We need } y(t)$$

$$v = \frac{dy}{dt} = b \quad \int 1 dy = \int b dt \quad y = y_0 + b \cdot t = y_0 + 2 \cdot t$$

Then
$$x - x_0 = \int_0^t a \cdot (y_0 + b \cdot t)^2 dt \quad x = x_0 + a \cdot \left(y_0^2 \cdot t + b \cdot y_0 \cdot t^2 + \frac{b^2 \cdot t^3}{3} \right)$$

Hence, with $x_0 = 1 \quad y_0 = 4$
$$x = 1 + 16 \cdot t + 8 \cdot t^2 + \frac{4}{3} \cdot t^3 \quad \text{At } t = 1 \text{ s} \quad x = 26.3 \cdot \text{m}$$

$$y = 4 + 2 \cdot t \quad y = 6 \cdot \text{m}$$

For particle that passed through (-3,0) at $t = 1$
$$\int 1 dy = \int b dt \quad y = y_0 + b \cdot (t - t_0)$$

$$x - x_0 = \int_{t_0}^t a \cdot (y_0 + b \cdot t)^2 dt \quad x = x_0 + a \cdot \left[y_0^2 \cdot (t - t_0) + b \cdot y_0 \cdot (t^2 - t_0^2) + \frac{b^2}{3} \cdot (t^3 - t_0^3) \right]$$

Hence, with $x_0 = -3, y_0 = 0$ at $t_0 = 1$
$$x = -3 + \frac{4}{3} \cdot (t^3 - 1) = \frac{1}{3} \cdot (4 \cdot t^3 - 13) \quad y = 2 \cdot (t - 1)$$

Evaluating at $t = 3$
$$x = 31.7 \cdot \text{m} \quad y = 4 \cdot \text{m}$$

This is a steady flow, so pathlines, streamlines and streaklines always coincide